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# Nonlinear transient thermal stress analysis of thick-walled FGM cylinder with temperature-dependent material properties

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**Abstract** In this paper, the problem of generalized thermoelasticity in a thick-walled FGM cylinder with one relaxation time is presented. The material properties are taking as function of temperature and graded in the radial. Due to the nonlinearity of the governing equations, finite element method is adopted to solve such problem. Both the inner and outer curved surfaces of the cylinder are stress free while the inner surface is subjected to thermal shock, while the outer surface is thermally isolated. The effects of temperature-dependent properties, volume fraction parameter and the thermal relaxation time on the physical quantities behavior are evaluated. Results confirm the efficiency of the present algorithm and reveal the significant effects of the temperature-dependent of the material properties.

**Keywords** Generalized thermoelasticity · Relaxation time · Thick-walled FGM cylinder · Temperature-dependent · Finite element method

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## 1 Introduction

A new class of intelligent materials, called functionally graded materials (FGMs), has been rapidly developed and used in engineering applications. FGMs are important non-homogeneous materials designed to work in a high temperature environment. Usually, these materials are made from a mixture of ceramic and metal. FGMs are mainly employed to withstand elevated temperatures and severe thermal gradients. Low thermal conductivity, low coefficients of thermal expansion, core ductility, and smooth stress distribution have enabled the FGMs to withstand higher thermal and mechanical.

The theory of coupled thermoelasticity was formulated by Biot [1] to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. However, both theories share the second shortcoming, because the heat equation for the coupled theory is of a mixed parabolic/hyperbolic type. Lord and Schulman [2] obtained a wave-type heat equation by postulating a new law of heat coeducation to replace the Fourier's classical law. This new law contains the heat flux vector as well as its derivative. It contains also a new constant that acts as a relaxation time introduced the theory of thermoelasticity with one relaxation time. This theory was extended by Dhaliwal and Sherief [3] to include the effect of anisotropic behaviour and the presence of heat source. Another well established generalized thermoelasticity theory is due to Green and Lindsay [4]. This theory is also referred as the temperature-rate dependent theory of thermoelasticity and takes into account two relaxation times. Later on, another three new formulations of thermoelasticity based on entropy balance inequality have been proposed by Green and Naghdi [5, 6].

Researchers presented so far in the transient heat transfer analysis of the FGM cylinders are very limited and almost all of them have ignored the temperature-dependency of the material properties. Reddy and Chin [7] and Praveen et al. [8] have developed Lagrangian finite element formulations to study the pseudo-dynamic thermoelastic responses of the functionally graded cylinders. Obata and Noda [9] analyzed the steady thermal stresses in a hollow circular cylinder and a hollow sphere of a functionally gradient material. Shao and Ma [10] investigated the thermo-mechanical stresses in functionally graded circular hollow cylinder with linearly increasing boundary temperature. Abbas and Zenkour [11] studied the LS model on electro-magneto-thermo-elastic response of an infinite functionally graded cylinder by finite element method. Most of the theoretical investigations reported so far have not taken into account the temperature dependence for the material properties. Therefore, those results in general are only adequate for small change of temperature in an FGM or the variation of material properties against temperature being insignificant. To accurately describe thermo-mechanical behaviors of FGMs, temperature dependence on the material properties should be considered. Recently Shariyat and his colleagues [12–16] introduce a transient thermal analysis taking the temperature-dependency of the material properties into account. Azadi and Azadi [17] studied the Nonlinear transient heat transfer and thermoelastic analysis of thick-walled FGM cylinder with temperature-dependent material properties using Hermitian transfinite element.

The objective of this paper is to study the nonlinear transient thermal stress of temperature-dependent thick-walled FGM cylinders subjected to thermal loading is presented. The finite element solution procedure is adopted to extract the results from the highly nonlinear governing equations. Numerical results based on the assumptions of temperature-dependent and temperature-independent of the material properties are compared with each other and effects of various parameters are also studied.

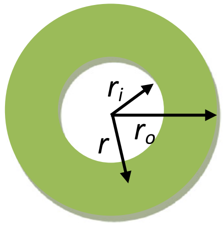

**Fig. 1** FGM thick-walled cylinder

## 2 The governing equations

Geometric parameters of the FGM thick-walled cylinder are shown in Fig. 1. The FGM thick-walled cylinder is assumed to be made of a mixture of two constituent materials so that the inner layer $(r = r_i)$ of the cylinder is ceramic-rich, whereas the external layer $(r = r_o)$ is metal-rich.

The material properties of the FGM thick-walled cylinder are assumed to be function of the volume fraction of the constituent materials. The functionally graded between the physical properties and the radial direction $r$ for ceramic and metal FGM thick-walled cylinder is given by.

$$P = P_m + (P_c - P_m)\left(\frac{r_o - r}{r_o - r_i}\right)^N, \tag{1}$$

where $P_c$ and $P_m$ are the corresponding properties of ceramic (inner layer) and metal (outer layer), respectively. The parameter $N$ is the volume fraction exponent which takes positive real values. For the FGM thick-walled cylinder constituents, i.e. ceramic and metal, the material properties are assumed to be temperature-dependent and a typical property $P$ of them can be expressed as a function of temperature as [18].

$$P = P_0\left(1 + P_1T + P_2T^2 + P_3T^3\right), \tag{2}$$

where $T$ is the absolute temperature, $P_0, P_1, P_2$ and $P_3$ are some material constants. From Eqs (1) and (2) one may conclude that

$$\begin{aligned} P = {} & P_{0c}\left(1 + P_{1c}T + P_{2c}T^2 + P_{3c}T^3\right)V_c \\ & + P_{0m}\left(1 + P_{1m}T + P_{2m}T^2 + P_{3m}T^3\right)V_m, \end{aligned} \tag{3}$$

where $V_m$ and $V_c$ are the volume fractions of the metallic and the ceramic constituent materials:

$$V_c = \left(\frac{r_o - r}{r_o - r_i}\right)^N, V_m = 1 - V_c. \tag{4}$$

The governing equation of motion of the axisymmetric cylinder in the absence of the body forces is

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}=\rho(r,T)\frac{\partial^2 u}{\partial t^2}, \tag{5}$$

where $\rho$ is the material density of the cylinder and it is also considered to be a function of the radius $r$ and the absolute temperature $T$, The mechanical stress components $\sigma_r$and $\sigma_\theta$ are given, respectively, by

$$\sigma_r=\lambda(r,T)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+2\mu(r,T)\frac{\partial u}{\partial r}-\gamma(r,T)T, \tag{6}$$

$$\sigma_\theta=\lambda(r,T)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+2\mu(r,T)\frac{u}{r}-\gamma(r,T)T, \tag{7}$$

where $u=u(r,t)$ is the radial displacement, $t$ is the time, $\lambda(r,T)$ and $\mu(r,T)$ are Lamé's coefficients, $\gamma(r,T)=[3\lambda(r,T)+2\mu(r,T)]\alpha(r,T)$ is the stress-temperature modulus, in which $\alpha(r,T)$ is the linear thermal expansion.

The heat conduction equation in the presence of one relaxation time can be written in the form.

$$\begin{aligned}&\frac{\partial^2 T}{\partial r^2}+\left(\frac{1}{k(r,T)}\frac{\partial k(r,T)}{\partial r}+\frac{1}{r}\right)\frac{\partial T}{\partial r}\\&=\frac{\rho(r,T)c_e(r,T)}{k(r,T)}\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)T\\&\quad+\frac{T_0\gamma(r,T)}{k(r,T)}\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right),\end{aligned} \tag{8}$$

where $k(r,T)$ is the thermal conductivity, $c_e(r,T)$ is the specific heat, $\tau_o$ is the relaxation time and $T_0$ is the reference temperature. Generally, this study assumes that $\lambda,\mu,\rho,k,c_e$ and $\gamma$ of the FGM thick-walled cylinder change continuously through the radial direction of the cylinder and obey the gradation relation given in Eq. (1) with the expression as a function of temperatures in Eq. (2), i.e. the material properties are radius and temperature-dependent. Therefore, the resulted system of governing equations is highly nonlinear. Derivatives of the material properties may be calculated based on Eqs. (3) and (4) as.

$$\begin{aligned}\frac{\partial P}{\partial r}&=\frac{\partial P_m}{\partial r}+\left(\frac{\partial P_c}{\partial r}-\frac{\partial P_m}{\partial r}\right)\left(\frac{r_o-r}{r_o-r_i}\right)^N\\&\quad-\frac{N(P_c-P_m)}{r_o-r_i}\left(\frac{r_o-r}{r_o-r_i}\right)^{N-1}\end{aligned} \tag{9}$$

with

$$\begin{aligned}&P_m=P_{0m}\left(1+P_{1m}T+P_{2m}T^2+P_{3m}T^3\right),\\&P_c=P_{0c}\left(1+P_{1c}T+P_{2c}T^2+P_{3c}T^3\right),\\&\frac{\partial P_m}{\partial r}=P_{0m}\left(P_{1m}+2P_{2m}T+3P_{3m}T^2\right)\frac{\partial T}{\partial r},\\&\frac{\partial P_c}{\partial r}=P_{0c}\left(P_{1c}+2P_{2c}T+3P_{3c}T^2\right)\frac{\partial T}{\partial r},\end{aligned} \tag{10}$$

## 3 Solution of the problem

In what follows, the effect of material properties variation of the FGM thick-walled cylinder should be taken into account. Introducing the following dimensionless variables may be simplifying the solving process:

$$\begin{aligned}(r',u',r_i',r_o')&=\frac{c}{\chi_{0m}}(r,u,r_i,r_o),\quad (t',\tau_o')\\&=\frac{c^2}{\chi_m}(t,\tau_o),\quad T'=\frac{T}{T_0},(\sigma_r',\sigma_\theta')\\&=\frac{1}{\lambda_{0m}+2\mu_{0m}}(\sigma_r,\sigma_\theta),\quad c^2\\&=\frac{\lambda_{0m}+2\mu_{0m}}{\rho_{0m}},\ \chi_{0m}=\frac{k_{0m}}{\rho_{0m}c_{e0m}}.\end{aligned} \tag{11}$$

The substituting of the aid of the dimensionless variables which given by Eq. (11) in the above equations, after dropping the prime ($'$) for convenience, produces the governing equations for the FGM thick-walled cylinder as follows:

$$\begin{aligned}&\frac{\partial}{\partial r}\left[\bar\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+2\bar\mu\frac{\partial u}{\partial r}-\bar\gamma T\right]\\&+2\bar\mu\left(\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)=\bar\rho\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{12}$$

$$\begin{aligned}&\frac{\partial^2 T}{\partial r^2}+\left(\frac{1}{\bar k}\frac{\partial \bar k}{\partial r}+\frac{1}{r}\right)\frac{\partial T}{\partial r}=\frac{1}{\bar\chi}\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)T\\&+\bar\varepsilon\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right),\end{aligned} \tag{13}$$

$$\sigma_r=\bar\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+2\bar\mu\frac{\partial u}{\partial r}-\bar\gamma T, \tag{14}$$

$$\sigma_\theta=\bar\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+2\bar\mu\frac{u}{r}-\bar\gamma T, \tag{15}$$

where the functions $\bar\lambda,\bar\mu,\bar\gamma,\bar k,\bar\chi,\bar\rho$ and $\bar\varepsilon$ may be given by